Local Dirac's condition on the existence of 2-factor

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Abstract

Let G be simple graph. For a vertex $u \in V(G)$, denote by $M_2(u)$ the set of vertices v with distance at most 2 away from u. We say that graph G satisfies the *local Dirac's condition* if for every vertex $u \in V(G)$, its degree $d(u)$ satisfies $d(u) \geq \frac{|M_2(u)|}{2}$.

It was conjectured that a connected graph G on at least three vertices satisfies the local Dirac's condition, then G is Hamiltonian. However, Asratian et al. disproved this conjecture. In this paper, we show that if a connected graph G on at least three vertices satisfies the local Dirac's condition, then it contains a 2-factor. Furthermore, this result is shown to be the best possible.

Keywords: local Dirac's condition, 2-factor, barrier

1. Introduction

In this paper, we consider simple graphs, that is, finite graphs without multiple edges and loops. For notation and terminology not defined here, readers are referred to [7]. A graph is *Hamiltonian* if it contains a spanning cycle. Determining whether a graph is Hamiltonian is one of few fundamental problems in graph theory. There are several well-known sufficient conditions for a graph to

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be Hamiltonian, such as Dirac's Theorem [10], Ore's Theorem [12] and Chvátal-Erdős Theorem [9]. These sufficient conditions involves some global parameters of a graph: the order, degrees of vertices, the independence number and the connectivity.

Inspired by the above global conditions, there are some analogues [2, 3, 4, 5, 8, 13] under the local criteria. Let G be a graph. For any two vertices $u, v \in V(G)$, we use $d(u, v)$ to denote the distance between u and v in G which is the minimum length of a path with the ends u and v. For a vertex $u \in V(G)$ and a nonnegative integer k, let $M_k(u)$ denote set S of vertices v such that $d(u, v) \leq k$. By definition, $M_0(u) = \{u\}$ and $M_1(u) = N(u) \cup \{u\}$ is the closed neighborhood of u. Diract's theorem asserts that a graph of order $n \geq 3$ is Hamiltonian if minimum degree $\delta(G) \geq n/2$. For each vertex $v \in V(G)$, let $d(v)$ denote the degree of u. Asratian and Khachatryan [2] generalized this result as follows.

Theorem 1.1 (Asratian and Khachatryan)**.** *Let G be a connected graph with at least three vertices. If $d(u) \geq \frac{|M_3(u)|}{2}$ for each vertex $u \in V(G)$, then G is Hamiltonian.*

Readers are referred to [3, 5, 11, 14, 16] for more local conditions on Hamiltonian graphs. We say a graph G satisfies the *local Dirac's condition* if $d(u) \geq \frac{|M_2(u)|}{2}$ for each vertex $u \in V(G)$. Asratian et al. [5] constructed a 2-connected graph G on at least three vertices that satisfies the local Dirac's condition but is not Hamiltonian. The graph in Fig.1 satisfies the local Dirac's condition but is not 2-connected. Consequently, it is not 1-tough.

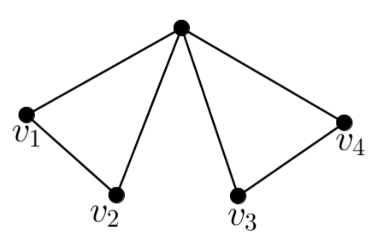

Figure 1: A graph G without a 2-factor.

In this paper, we prove the local Dirac's condition is a sufficient for a graph to contain a 2-factor.

Theorem 1.2. *Let G be a connected graph with at least three vertices. If $d(u) \geq \frac{|M_2(u)|}{2}$ for each vertex $u \in V(G)$, then G contains a 2-factor.*

The graph G in Fig.2 contains no 2-factor and $d(v_i) = \frac{|M_2(v_i)|-1}{2}$ for each vertex v_i in $G, i \in \{1, 2, 3, 4\}$. Thus the bound in Theorem 1.2 is sharp.

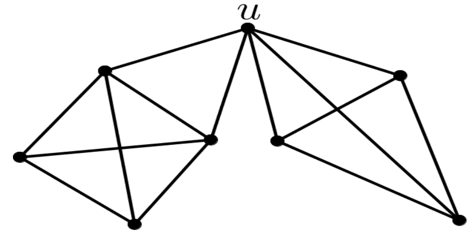

Figure 2: A graph with a cut vertex u satisfying the local Dirac's condition.

In the following, we introduce some notation will be used in this paper. Let G be a graph and $v \in V(G)$. Denote by $N_G(v)$ the set of neighbors of v in G, and denote by $N_G[v]$ the set $N_G(v) \cup \{v\}$. If there is no confusion, we use $N(v)$ and $N[v]$ to denote $N_G(v)$ and $N_G[v]$, respectively. For a vertex set $S \subseteq V(G)$, let $N_S(v) = S \cap N_G(v), d_S(v) = |N_S(v)|$. Let $d(v)$ denote $d_G(v)$ for brevity if there is no confusion. For a graph G with $A, B \subseteq V(G)$, let $e_G(A, B)$ denote the number of edges with one end in A and the other end in B, and let $e_G(v, B)$ denote $e_G(\{v\}, B)$.

Let $N_2(u) = M_2(u) \setminus N[u]$ and $d_2(u) = |N_2(u)|$. In a subgraph H of a graph G, v is called an *interior vertex* of H if $N[v] \subseteq V(H)$. Let $\alpha(G)$ and $\kappa(G)$ denote the independence number and the connectivity of a graph G, respectively.

2. Preliminaries

A multigraph is a finite graph that may contain parallel edges but no loops. Let G be a multigraph and $S, T \subseteq V(G)$ be two disjoint vertex sets. A component C of $G-(S \cup T)$ C is called an *odd component* (resp., *even component*) with respect to (S,T) if $e_G(C,T) \equiv 1 \pmod 2$ (resp., $e_G(C,T) \equiv 0 \pmod 2$). Let $\mathcal{H}_G(S,T)$ denote the set of the odd components of $G-(S\cup T)$ and $h_G(S,T) = |\mathcal{H}_G(S,T)|$; moreover, let $\delta_G(S,T) = 2|S| - 2|T| + \sum\limits_{x\in T} d_{G-S}(x) - h_G(S,T)$. The following sufficient and necessary condition on the existence of a 2-factor is derived from Tutte's f-factor theorem in [15].

Theorem 2.1. ([15]) A multigraph G contains a 2-factor if and only if $\delta_G(S,T) \geq 0$ for every $S,T \subseteq V(G)$ with $S\cap T = \emptyset$.

Following the definition of $\delta_G(S,T)$, we can verify that $\delta_G(S,T) \equiv 0 \pmod 2$ for every $S,T \subseteq V(G)$ with $S\cap T = \emptyset$. By Theorem 2.1, if a graph G contains no 2-factor, then G has an ordered pair (S,T) with $S\cap T = \emptyset$ and $\delta_G(S,T) \leq -2$.. We call an ordered pair (S,T) a *barrier* if $S\cap T = \emptyset$ and $\delta_G(S,T) \leq -2$. A barrier (S,T) is called a *minimal barrier* if $|S\cup T|$ is minimized among all the barriers of G. The following result gives the characterizations of a minimal barrier, in which (1)-(3) are obtained from [1] and (4)-(5) are from [8].

Lemma 2.1. ([1, 8]) Let G be a graph without a 2-factor. If (S,T) is a minimal barrier of G, then

(1) T is independent;

(2) if C is an even component with respect to (S,T), then $e_G(T,C) = 0$;

(3) if C is an odd component with respect to (S,T), then $e_G(v,C) \leq 1$ for every $v \in T$;

(4) for every $v \in S, |\{C \in \mathcal{H}_G(S,T) : e_G(v,C) \geq 1\}| + e_G(v,T) \geq 4$;

(5) $|T| > |S| + \sum_{k\geq 1} k \cdot |\mathcal{C}_{2k+1}|$, where $\mathcal{C}_{2k+1}$ is the union of components C in $\mathcal{H}_G(S,T)$ with $e_G(C,T) = 2k+1, k \geq 0$.

The following result is an important tool in the proof of our main result.

Lemma 2.2. *Let $G[X,Y]$ be a bipartite graph without isolated vertices. If one of the following two conditions holds for each vertex $y \in Y$) $xy \in E(G)$, then $|X| \leq |Y|$.*

(i) $d(x) \geq d(y)$ for every $x \in N(y)$.

(ii) $d(y) \geq 3$, there is a neighbor $x_0 \in N(y)$ such that $d(x_0) = d(y) - 1$ and $d(y) \geq 3$ and $d(x) \geq d(y) + 1$ for ever vertex in $x \in N(y) \setminus \{x_0\}$.

Proof. For each edge $xy \in E(G)$ with $x \in X$ and $y \in Y$, we assign as weight $w(xy) = \frac{1}{d(x)}$. Then,

$$\sum_{e \in E(G)} w(e) = \sum_{x \in X} \sum_{y \in N(x)} \frac{1}{d(x)} = \sum_{x \in X} 1 = |X|$$

For each vertex $y \in Y$, if (i) holds, then $\sum\limits_{x \in N(y)} w(xy) = \sum\limits_{x \in N(y)} \frac{1}{d(x)} \leq \frac{|N(y)|}{d(y)} = 1$. If (ii) holds, then $\sum\limits_{x \in N(y)} w(xy) \leq \sum\limits_{x \in N(y) \setminus \{x_0\}} \frac{1}{d(y)+1} + \frac{1}{d(y)-1} = \frac{d(y)-1}{d(y)+1} + \frac{1}{d(y)-1} \leq 1$ since $d(y) \geq 3$. Hence, in both case we have $\sum_{x \in N(y)} w(xy) \leq 1$, which in turn shows that

$$\sum_{e \in E(G)} w(e) = \sum_{y \in Y} \sum_{x \in N(y)} \frac{1}{d(x)} \leq \sum_{y \in Y} 1 = |Y|.$$

Therefore, $|X| = \sum_{e \in E(G)} w(e) \leq |Y|$. □

AAA Note that $M_2(v) = N[v] \cup N_2(v)$ and $|M_2(v)| = d(v) + d_2(v) + 1$ for each vertex v in a graph. Then, by simple calculation, we can obtain the following result.

Remark 1. *If G satisfies the local Dirac's condition, then $d(v) \geq d_2(v) + 1$ for each vertex v of G.*

3. Proof of Theorem 1.2

Suppose on the contrary, there exists a connected graph G with at least three vertices satisfying the local Dirac's condition, but G contains no 2-factor. Let $E(\mathcal{H}_G(S,T))$ denote the union of the edge sets of all components in $\mathcal{H}_G(S,T)$, and for a vertex set W of $S \cup T$, let $e_G(W, \mathcal{H}_G(S,T))$ denote the number of edges between W and all the components in $\mathcal{H}_G(S,T)$. Then, by Theorem 2.1, we choose a barrier (S,T) of G such that

(1) (S,T) is a minimal barrier of G;

(2) subject to (1), $|E(\mathcal{H}_G(S,T))|$ is maximized;

(3) subject to (1) and (2), $e_G(S,\mathcal{H}_G(S,T))$ is maximized.

Claim 1. $\delta(G)\geq 2$.

Proof. We have either $N[v]=V(G)$ or $d_2(v)\geq 1$ for each vertex $v\in V(G)$. If $N[v]=V(G)$, then $d(v)\geq |V(G)|-1\geq 2$. If $d_2(v)\geq 1$, then $d(v)\geq d_2(v)+1\geq 2$. □

Claim 2. *For each $v\in T$, if $N_C(v)\neq\emptyset$ for some component $C\in\mathcal{C}_1$ with $|C|=1$, then $d_{G-S}(v)=1$.*

Proof. Suppose on the contrary, there exists a vertex $v\in T$ and a component $C\in\mathcal{C}_1$ with $|C|=1$ and $d_{G-S}(v)\geq 2$. Let $C=\{u\}$. By Lemma 2.1 (1)-(3), there are $d_{G-S}(v)$ components of $\mathcal{H}_G(S,T)$, in each one of which v has exactly one neighbor. Thus $e_G(\{v\},\mathcal{H}_G(S,T))=d_{G-S}(v)\geq 2$. Let $T':=(T\cup\{u\})\setminus\{v\}$. Clearly, $|S\cup T|=|S\cup T'|$. We have $h_G(S,T')=h_G(S,T)-d_{G-S}(v)+1$ by Lemma 2.1 (2)-(3), and $\sum_{w\in T'}d_{G-S}(w)=\sum_{w\in T}d_{G-S}(w)-d_{G-S}(v)+1$. Thus, $\delta_G(S,T')=\delta_G(S,T)$. Since $|C|=1$ and $d_{G-S}(v)\geq 2$, we have $|E(\mathcal{H}_G(S,T'))|>|E(\mathcal{H}_G(S,T))|$, a contradiction to the choice of (S,T). Thus $d_{G-S}(v)=1$. □

For each vertex $v\in T$, we define a mapping f_v from $\mathcal{H}_G(S,T)$ to $\mathcal{P}(N_2(v))$ such that $f_v(C)=N_2(v)\cap V(C)$ for $C\in\mathcal{H}_G(S,T)$, where $\mathcal{P}(N_2(v))=\{S:S\subseteq N_2(v)\}$. Clearly, $f_v(C)\cap f_v(C')=\emptyset$ if C and C' are two distinct components in $\mathcal{H}_G(S,T)$.

By Claim 2 and Lemma 2.1 (3), it is easy to obtain the following result.

Claim 3. *For each vertex $v\in T$, if $N_C(v)\neq\emptyset$ for some component $C\in\bigcup_{k\geq 1}\mathcal{C}_{2k+1}$, then $|C'|\geq 2$ and $f_v(C')\neq\emptyset$ for each component C' in $\mathcal{C}_1$ with $N_{C'}(v)\neq\emptyset$.*

By Lemma 2.1 (3), each vertex $v\in T$ has at most one neighbor in each component of $\mathcal{H}_G(S,T)$. We have the following result.

Claim 4. *For any vertex $v \in T$ and any component $C \in \mathcal{H}_G(S,T)$ with $|C| \geq 2$, if $N_C(v) \neq \emptyset$, then $f_v(C) \neq \emptyset$.*

Claim 5. *Given an edge uv with $v \in T, u \in V(C)$, and $C \in \bigcup_{k\geq 1} \mathcal{C}_{2k+1}$, if $d_{G-\mathcal{C}_1}(v) \leq 2$, then $N_T(u) = \{v\}, d_C(u) = 1$ and $d_S(v) = 1$.*

Proof. By Lemma 2.1 (3), $N_C(v) = \{u\}$. Let $\mathcal{C}_1' = \{C \in \mathcal{C}_1 : N_C(v) \neq \emptyset\}$. By Claim 3, $|C'| \geq 2$ and $f_v(C') \neq \emptyset$ for each component C' in $\mathcal{C}_1'$ provided $\mathcal{C}_1' \neq \emptyset$. Thus, $|\mathcal{C}_1'| \leq \sum_{C' \in \mathcal{C}_1'} |f_v(C')| \leq d_2(v)$. We have $d_2(v) \leq |\mathcal{C}_1'| + 1$. Otherwise, $d_2(v) \geq |\mathcal{C}_1'| + 2$, and then $d(v) \geq d_2(v) + 1 \geq |\mathcal{C}_1'| + 3$ by Lemma 1, which implies $d_{G-\mathcal{C}_1}(v) \geq 3$, giving a contradiction.

Suppose $|C| = 1$, i.e., $C = \{u\}$. Then, $d_T(u) \geq 3$ since $C \in \bigcup_{k\geq 1} \mathcal{C}_{2k+1}$. Note that $N_T(u) \setminus \{v\} \subseteq N_2(v)$ by Lemma 2.1 (1). It follows that $d_2(v) \geq \sum_{C' \in \mathcal{C}_1'} |f_v(C')| + (d_T(u) - 1) \geq |\mathcal{C}_1'| + (d_T(u) - 1) \geq |\mathcal{C}_1'| + 2$, giving a contradiction. Thus, $|C| \geq 2$ and then $|f_v(C)| \geq 1$ by Claim 4, which implies $d_2(v) \geq |\mathcal{C}_1'| + |f_v(C)| \geq |\mathcal{C}_1'| + 1$. Note that $d_C(u) = |f_v(C)|$. If $|f_v(C)| \geq 2$ or $N_T(u) \setminus \{v\} \neq \emptyset$, then we have $d_2(v) \geq |\mathcal{C}_1'| + 2$, giving a contradiction. Thus, $N_T(u) = \{v\}$ and $d_C(u) = 1$. Moreover, we have $d_2(v) = |\mathcal{C}_1'| + 1$ and hence $d(v) \geq |\mathcal{C}_1'| + 2$ by Lemma 1, which implies $d_{G-\mathcal{C}_1}(v) = 2$ by $d_{G-\mathcal{C}_1}(v) \leq 2$. Suppose $N_S(v) = \emptyset$. Then, by $d_{G-\mathcal{C}_1}(v) = 2$ and Lemma 2.1 (1), there is a component $C' \in (\bigcup_{k\geq 1} \mathcal{C}_{2k+1}) \setminus \{C\}$ with $N_{C'}(v) \neq \emptyset$. As the preceding proof for C, $|C'| \geq 2$ and $|f_v(C')| = 1$. It follows that $d_2(v) \geq |\mathcal{C}_1'| + |f_v(C')| + |f_v(C)| = |\mathcal{C}_1'| + 2$, giving a contradiction. Thus $N_S(v) \neq \emptyset$, and $d_S(v) = 1$ by $d_{G-\mathcal{C}_1}(v) = 2$. □

Claim 6. *For $v \in T$, if $N_C(v) \neq \emptyset$ for some $C \in \bigcup_{k\geq 1} \mathcal{C}_{2k+1}$, then $d_{G-\mathcal{C}_1}(v) \geq 3$.*

Proof. Denote by u the neighbor of v in C. Suppose on the contrary, $d_{G-\mathcal{C}_1}(v) \leq 2$. Then, $d_S(v) = 1$ by Claim 5. Let $N_S(v) = \{w\}$. Clearly, $w \neq u$. Since $d_{G-\mathcal{C}_1}(v) \leq 2$, we have $d_{G-\mathcal{C}_1}(v) = 2$ by $\{w, u\} \subseteq N(v)$. By Claim 5, $d_C(u) = 1$, which implies $|C| \geq 2$ and $|f_v(C)| = 1$. Let $N_C(u) = \{u_1\}$. Then, $f_v(C) = \{u_1\}$. Suppose $|N_T(w)| \geq 2$ and let $w_1 \in N_T(w) \setminus \{v\}$. Then, $w_1 \in N_2(v)$ by Lemma 2.1 (1). Let $\mathcal{C}_1' = \{C \in \mathcal{C}_1 : N_C(v) \neq \emptyset\}$. By Claim 3, $f_v(C') \neq \emptyset$ for each component C' in $\mathcal{C}_1'$ provided $\mathcal{C}_1' \neq \emptyset$. Clearly, $u_1 \neq w_1$ and $\{u_1, w_1\} \subseteq N_2(v)$. Moreover, $\{u_1, w_1\} \cap f_v(C') = \emptyset$ for each $C' \in \mathcal{C}_1'$. It follows that $d_2(v) \geq |\mathcal{C}_1'|+2$, which implies $d_{G-\mathcal{C}_1}(v) \geq 3$, giving a contradiction. Thus $N_T(w) = \{v\}$.

By Lemma 2.1 (4), there are at least three components of $\mathcal{H}_G(S,T)$ in which w has a neighbor. Suppose $N_{C^*}(w) \neq \emptyset$ and $N_{C^*}(v) = \emptyset$ for a component $C^* \in \mathcal{H}_G(S,T)$. Let $w^* \in N_{C^*}(w)$. Then, $w^* \in N_2(v)$. Clearly, $w^* \neq u_1$ and $\{w^*, u_1\} \subseteq N_2(v)$. Moreover, $\{u_1, w^*\} \cap f_v(C') = \emptyset$ for each $C' \in \mathcal{C}_1'$. Thus $d_2(v) \geq |\mathcal{C}_1'|+2$, and then $d_{G-\mathcal{C}_1}(v) \geq 3$, giving a contradiction. Thus $N_{C^*}(v) \neq \emptyset$ for each component $C^* \in \mathcal{H}_G(S,T)$ with $N_{C^*}(w) \neq \emptyset$. It follows that there are at least three components of $\mathcal{H}_G(S,T)$ in which v has a neighbor. Moreover, by $d_S(v) = 1$, we have $d_G(v) \geq 4$, which implies $|\mathcal{C}_1'| \geq 2$ by $d_{G-\mathcal{C}_1}(v) = 2$. Suppose C_1, C_2 are two distinct components in $\mathcal{C}_1'$ and $v_i \in f_v(C_i), i \in \{1,2\}$ by Claim 3. Clearly, $\{v_1, v_2\} \subseteq N_2(v)$. Recall that $f_v(C) = \{u_1\}$. Since $C \in \bigcup_{k\geq 1} \mathcal{C}_{2k+1}$ and $d_T(u) = d_C(u) = 1$ by Claim 5, there is some vertex $u' \in V(C) \setminus \{u\}$ with $N_T(u') \neq \emptyset$, which implies $N(u_1) \setminus N(u) \neq \emptyset$. Thus there exists a vertex $u^* \in N(u_1)$ with $u^* \in N_2(u)$. Clearly, $\{u^*\} \cap N_{C_i}(v) = \emptyset, i = 1,2$, and hence $d_2(u) \geq 3$. Thus $d(u) \geq d_2(u) + 1 \geq 4$, which implies $d_S(u) \geq 2$ by $d_C(u) = 1$ and $d_T(u) = 1$. Let $u_2 \in N_S(u) \setminus \{w\}$. By $N_S(v) = \{w\}$, we have $u_2 \in N_2(v)$. Clearly, $u_1 \neq u_2$ and $\{u_1, u_2\} \cap f_v(C') = \emptyset$ for each $C' \in \mathcal{C}_1'$. Thus, $d_2(v) \geq |\mathcal{C}_1'| + |\{u_1, u_2\}| = |\mathcal{C}_1'| + 2$, and then $d_{G-\mathcal{C}_1}(v) \geq 3$, giving a contradiction. $\square$

Let H be the resulting graph obtained by doing the following operations on G:

(1) Remove all the even components;

(2) Remove all the components in $\mathcal{C}_1$;

(3) Remove all the edges in $G[S]$;

(4) For each component $C \in \bigcup_{k\geq 1} \mathcal{C}_{2k+1}$, suppose $N_T(C) = \{v_0^C, v_1^C, \cdots, v_{2k}^C\}$. Firstly, replace C by an independent set $U^C = \{u_1^C, u_2^C, \cdots, u_k^C\}$. Secondly, join u_i^C to v_{2i-1}^C and v_{2i}^C, respectively, and moreover, join u_1^C to v_0^C, $1 \leq i \leq k$.

Clearly, the vertices in $S \cup T$ of G are not changed in H, and we still use S and T to denote the two vertex sets in H. Since T is an independent set in G by Lemma 2.1 (1), by the above operations, H is a bipartite graph. In the following proof, let $H = H[Y, T]$ and $Y_1 = Y \setminus S$, where $Y = S \cup (\bigcup_{k \geq 1} \bigcup_{C \in \mathcal{C}_{2k+1}} U^C)$. By the above operations, we can obtain the following two results.

Claim 7. $|Y| = |S| + \sum_{k \geq 1} k \cdot |\mathcal{C}_{2k+1}|$.

Claim 8. *$d_H(y) \leq 3$ for each vertex $y \in Y_1$.*

Claim 9. *For each vertex $v \in T$, $d_H(v) = d_{G-\mathcal{C}_1}(v) \geq 1$.*

Proof. By Lemma 2.1 (1)-(2), $N_G(v) \subseteq S \cup (\bigcup_{k \geq 0} \mathcal{C}_{2k+1})$ for each vertex $v \in T$. Thus we have $d_H(v) = d_{G-\mathcal{C}_1}(v)$ from the operations on G. Suppose on the contrary that H contains an isolated vertex v in T. Then, $N_G(v) \subseteq \bigcup_{C \in \mathcal{C}_1} C$. Let $\mathcal{C}'_1$ denote the union of the components in $\mathcal{C}_1$, in which v has a neighbour. Then, $|\mathcal{C}'_1| \geq 2$ since $d_G(v) \geq 2$ by Claim 1 and Lemma 2.1 (3), and hence $|C| \geq 2$ by Claim 2, for each component $C \in \mathcal{C}'_1$. Moreover, each C in $\mathcal{C}'_1$ contains at least one vertex in $N_2(v)$ in G by Claim 4. Thus $d_2(v) \geq \sum_{C \in \mathcal{C}'_1} |f_v(C)| \geq |\mathcal{C}'_1|$, which implies $d_G(v) \geq |\mathcal{C}'_1| + 1$. It follows that $N_G(v)$ contains a vertex not in any component of $\mathcal{C}'_1$, a contradiction. □

Claim 10. *For any $v \in T$, if $N_G(v) \cap V(C) = \emptyset$ for each component $C \in \mathcal{C}_1$ with $|C| = 1$, then $d_H(v) \geq d_H(u)$ for each vertex $u \in N_H(v)$.*

Proof. Clearly, $N_H(v) \cap Y_1 \neq \emptyset$ if and only if $N_G(v)$ has a neighbor in some component of $\bigcup_{k \geq 1} \mathcal{C}_{2k+1}$. Suppose $N_G(v)$ has a neighbor in some component of $\bigcup_{k \geq 1} \mathcal{C}_{2k+1}$. Then, $d_{G-\mathcal{C}_1}(v) \geq 3$ by Claim 6, and hence $d_H(v) \geq 3 \geq d_H(y)$ by Claim 8 and Claim 9 for each $y \in Y_1 \cap N_H(v)$.

By the operations on G, $S \cap N_H(v) = S \cap N_G(v)$. Suppose $w \in S \cap N_H(v)$. Let $\mathcal{C}'_1 = \{C \in \mathcal{C}_1 : N_C(v) \neq \emptyset\}$. By the hypothesis of the claim, $|C'| \geq 2$ for each $C' \in \mathcal{C}'_1$ provided $\mathcal{C}'_1 \neq \emptyset$, and hence $|f_v(C')| \geq 1$ by Claim 4. Since $N_H(w) \subseteq T$, we have $N_H(w) \setminus \{v\} \subseteq N_2(v)$ by Lemma 2.1 (1). Clearly,

$(\bigcup_{C'\in\mathcal{C}_1'} f_v(C')) \cap (N_H(w)\setminus\{v\}) = \emptyset$. Then $d_G(v) \geq d_2(v)+1 \geq \sum_{C'\in\mathcal{C}_1'} |f_v(C')| + d_H(w) \geq |\mathcal{C}_1'| + d_H(w)$. Thus, $d_H(v) = d_{G-\mathcal{C}_1}(v) \geq d_H(w)$. $\square$

Claim 11. *For any vertex $v \in T$, if there exists a vertex $u \in N_H(v)$ with $d_H(u) > d_H(v)$, then $d_H(u) \geq 3, d_H(v) = d_H(u)-1$, and $d_H(v') \geq d_H(u)+1$ for each vertex $v' \in N_H(u)\setminus\{v\}$.*

Proof. By $d_H(u) > d_H(v)$ and Claim 10, v has a neighbor in some component $C \in \mathcal{C}_1$ with $|C| = 1$. Suppose $C = \{w\}$. Then, $N_G(v) \subseteq S \cup \{w\}$ by Claim 2, and so $u \in S$ and $d_H(v) = d_G(v) - 1$. Since $H = H[Y,T]$ is a bipartite graph, $N_H(u) \subseteq T$. By Lemma 2.1 (1), $(N_H(u)\setminus\{v\}) \subseteq N_2(v)$. Thus, $d_G(v) \geq d_2(v)+1 \geq d_H(u)$, which implies $d_H(v) = d_G(v)-1 \geq d_H(u)-1$. By $d_H(u) > d_H(v)$, we have $d_H(v) = d_H(u)-1$, which implies $N_2(v) = N_H(u)\setminus\{v\}$. Thus, $(N_G(u)\setminus T) \subseteq N_G(v)$, and u has at most w as a neighbor in the components of $\mathcal{H}_G(S,T)$. It follows that $d_H(u) = |N_G(u)\cap T| \geq 3$ by Lemma 2.1 (4). Since $N_2(v) = N_H(u)\setminus\{v\} \subseteq T$ and $N_G(w) \subseteq S\cup\{v\}$, we have $N_G[w] \subseteq N_G[v]$. We have $N_G[w] = N_G[v]$. Otherwise, $N_G(w)\setminus\{v\}$ is a proper subset of $N_G(v)\setminus\{w\}$, which implies $|e_G(w,S)| < |e_G(v,S)|$. Let $T' := (T\cup\{w\})\setminus\{v\}$, $C' := \{v\}$, and $\mathcal{H}_G(S,T') = (\mathcal{H}_G(S,T)\setminus\{C\})\cup\{C'\}$. By $|C| = 1$ and $d_{G-S}(v) = 1$, it is easy to see that $\delta_G(S,T') = \delta_G(S,T)$. By $|e_G(v,S)| > |e_G(w,S)|$, we have $e_G(S,\mathcal{H}_G(S,T')) > e_G(S,\mathcal{H}_G(S,T))$, giving a contradiction to the choice of (S,T). Thus, we have $w \in N(u)$ by $N_G[w] = N_G[v]$. Since $N_T(w) = \{v\}$, we have $\{w\}\cup(N_H(u)\setminus\{u'\}) \subseteq N_2(u')$ for each vertex $u' \in N_H(u)\setminus\{v\}$. Thus, $d_G(u') \geq d_2(u')+1 \geq d_H(u)+1$.

Let $u_1 \in N_H(u)\setminus\{v\}$. Suppose u_1 has no neighbor in any component of $\mathcal{C}_1$. Then $d_H(u_1) = d_G(u_1) \geq d_H(u)+1$ by $u_1 \in T$. Suppose $N_{C'}(u_1) \neq \emptyset$ for some component $C' \in \mathcal{C}_1$ with $|C'| = 1$. By $N_T(w) = \{v\}$ and $w \in N(u)$, we have $w \in N_2(u_1)$. Let $C' = \{w'\}$. Then, $N_{G-S}(u_1) = \{w'\}$ by Claim 2. Clearly, $w \neq w'$ and $N_G(w') \subseteq S\cup\{u_1\}$. Suppose there is a vertex $u_2 \in N_G(w')\setminus\{u_1\}$ with $u_1u_2 \notin E(G)$. Then, $u_2 \in S$ and hence $u_2 \neq w$. Thus $\{u_2,w\}\cup(N_H(u)\setminus\{u_1\}) \subseteq N_2(u_1)$ and $d_2(u_1) \geq d_H(u)+1$, which implies $d_G(u_1) \geq d_2(u_1)+1 \geq d_H(u)+2$. Since $N_{G-S}(u_1) = \{w'\}$, we have $d_H(u_1) = d_G(u_1)-1 \geq d_H(u)+1$.

Suppose $N_G(w') \subseteq N_G[u_1]$. Then, $N_G[w'] = N_G[u_1]$. Otherwise, $|e_G(u_1, S)| > |e_G(w', S)|$. Let $T^* := (T \cup \{w'\}) \setminus \{u_1\}$. As the preceding proof for v and w, we have $\delta_G(S, T^*) = \delta_G(S, T)$, and $e_G(S, \mathcal{H}_G(S, T^*)) > e_G(S, \mathcal{H}_G(S, T))$, giving a contradiction to the choice of (S, T). Thus $uw' \in E(G)$ by $u \in N_G(u_1)$, which implies $w' \in N_2(v)$, giving a contradiction with $N_2(v) = N_H(u) \setminus \{v\}$. Suppose $|C''| \geq 2$ for each component $C'' \in \mathcal{C}_1$ with $N_{C''}(u_1) \neq \emptyset$. Then, by Claim 4, $|f_{u_1}(C'')| \geq 1$ for each component $C'' \in \mathcal{C}_1$ with $N_{C''}(u_1) \neq \emptyset$. Note that $\bigcup_{C'' \in \mathcal{C}_1} f_{u_1}(C'') \cup (N_H(u) \setminus \{u_1\}) \cup \{w\} \subseteq N_2(u_1)$. Then $d_G(u_1) \geq \sum_{C'' \in \mathcal{C}_1} |f_{u_1}(C'')| + d_H(u) + 1$. Thus $d_H(u_1) \geq d_H(u) + 1$. □

By Claim 9, T contains no isolated vertex in H. Note that Y may contain some isolated vertex y in H if and only if $y \in S$ with $N_G(y) \cap T = \emptyset$. Let $Y' = N_H(T)$ and $H' := H[Y', T]$ be a subgraph of $H[Y, T]$. By Claim 10 and Claim 11, each edge in H' satisfies the hypothesis of Lemma 2.2, and hence $|T| \leq |Y'| \leq |Y|$ by Lemma 2.2. By Lemma 2.1 (5) and Claim 7, we have $|T| > |Y|$, giving a contradiction. Thus Theorem 1.2 is true.

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